

# DEFORMATION OF ALGEBRA FACTORISATIONS

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## Abstract

A general deformation theory of algebras which factorise into two subalgebras is studied. It is shown that the classification of deformations is related to the cohomology of a certain double complex reminiscent of the Gerstenhaber-Schack complex of a bialgebra.

## 1 Introduction

An algebra factorisation or a twisted tensor product is a unital algebra  $X$  over a field  $k$  together with two (unital) subalgebras  $B, A$  such that the map  $B \otimes A \rightarrow X$  given by multiplication is an isomorphism. In what follows we identify  $X$  with  $B \otimes A$  as a  $(B, A)$ -bimodule via this isomorphism and hence the algebra structure on  $X$  can be viewed as a twisting of the usual tensor product algebra. The algebra  $X$  consists of elements of the form  $x = \sum_i b_i \otimes a_i = \sum_i b_i a_i$ , where  $b_i \in B$  and  $a_i \in A$ . An algebra factorisation is denoted by  $X(B, A)$ . Algebra factorisations appear frequently in algebra and number theory. Examples include the tensor product and the braided tensor product algebras. Also the quaternions can be viewed as an algebra factorisation over the real numbers built on two copies of the complex

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numbers (cf. [2][3]). Of physical interest are algebra factorisations obtained by the quantisation of phase spaces. For example, the Heisenberg algebra is a factorisation built on the algebras generated by the momentum and position operators.

The theory of deformations of a twisted tensor product between *undeformed* algebras  $B$  and  $A$  was introduced in [4], where it has been shown that the Heisenberg algebra is such a deformation of the usual tensor product. The aim of this note is to give a cohomological interpretation of a general theory of deformations of algebra factorisations which allows for deformation of all the algebras  $A$ ,  $B$  and  $X$  entering the factorisation.

In [8][7][4] it is shown that factorisations are in one-to-one correspondence with linear maps  $\Psi : A \otimes B \rightarrow B \otimes A$  such that

$$\Psi(\mu_A \otimes B) = (B \otimes \mu_A)(\Psi \otimes A)(A \otimes \Psi), \quad \Psi(1_A \otimes B) = B \otimes 1_A, \quad (1)$$

$$\Psi(A \otimes \mu_B) = (\mu_A \otimes B)(B \otimes \Psi)(\Psi \otimes B), \quad \Psi(A \otimes 1_B) = 1_B \otimes A. \quad (2)$$

Here and below the following notation is used. For an algebra  $A$ , the identity map on  $A$  is denoted by  $A$ , the unit of  $A$ , viewed either as an element of  $A$  or as a map  $k \rightarrow A$  is denoted by  $1_A$  and the product map is denoted by  $\mu_A$ . For a given factorisation  $X(B, A)$  the map  $\Psi$  is given by  $ab = (\mu_X \circ \Psi)(a \otimes b)$ , and the equations (1), (2) simply express the associativity conditions:  $(aa')b = a(a'b)$ ,  $a(bb') = (ab)b'$  and the unit condition  $1_A b = b$ ,  $a 1_B = a$ , for all  $a, a' \in A$  and  $b, b' \in B$ . We will write  $\Psi(a \otimes b) = \sum_\nu b_\nu \otimes a^\nu$ , i.e,  $ab = \sum_\nu b_\nu a^\nu$ . All this means that the structure of an algebra factorisation  $X(B, A)$  is fully described by three maps: product in  $A$   $\mu_A : A \otimes A \rightarrow A$ , product in  $B$   $\mu_B : B \otimes B \rightarrow B$  and the twisting  $\Psi : A \otimes B \rightarrow B \otimes A$ .

A *deformation of an algebra factorisation*  $X(B, A)$  over  $k$  is an algebra factorisation  $X_t(B_t, A_t)$  over  $k[[t]]$  such that the algebras  $A_t$ ,  $B_t$  and  $X_t$  are algebra deformations of  $A$ ,  $B$  and  $X$  respectively. This means that each of the maps  $\mu_{A_t}$ ,  $\mu_{B_t}$  and  $\Psi_t$  corresponding to  $X_t(B_t, A_t)$  can be written as a formal power series

$$\mu_{A_t} = \mu_A + \sum_{i=1} t^i \mu_A^{(i)}, \quad \mu_{B_t} = \mu_B + \sum_{i=1} t^i \mu_B^{(i)}, \quad \Psi_t = \Psi + \sum_{i=1} t^i \Psi^{(i)}, \quad (3)$$

where  $\mu_A^{(i)} : A \otimes A \rightarrow A$ ,  $\mu_B^{(i)} : B \otimes B \rightarrow B$ ,  $\Psi^{(i)} : A \otimes B \rightarrow B \otimes A$ , and  $\mu_A$ ,  $\mu_B$  and  $\Psi$  describe the factorisation  $X(B, A)$ . This definition of a deformation of an algebra factorisation generalises the definition introduced in [4], where only the map  $\Psi$  was allowed to be deformed. The need for such a generalisation arises from the theory of quantum and coalgebra principal bundles [1]. As explained in [2] the structure of a classical principal bundle is encoded in the factorisation built on

the algebra of functions on the total space of a bundle and the group algebra of the structure group. The action of the structure group determines the twisting  $\Psi$ . Similarly the structure of a coalgebra principal bundle is encoded into an algebra factorisation, termed a *Galois factorisation* in [2]. In many cases the algebras on which this factorisation is built are deformations of their classical counterparts so that not only the twisting map  $\Psi$  but also the algebras  $A$  and  $B$  are deformed.

In this note we show that, similarly to the Gerstenhaber theory of deformation of algebras [5], there is a cohomological interpretation of deformations of algebra factorisations. Interestingly, such an interpretation uses the total cohomology of a certain double complex. The situation is therefore somewhat reminiscent of the Gerstenhaber-Schack theory of deformations of bialgebras [6]. This is not entirely surprising as there is a close relationship between algebra factorisations and *entwining structures* (cf. [1]). The latter can be seen as a generalisation of a bialgebra, and, from this point of view, a need for a double complex in the description of algebra factorisations should be expected. Deformation theory of entwining structures as well as the corresponding cohomology theory will be discussed elsewhere.

## 2 Construction of the cochain complex

The fact that  $X(B, A)$  is a factorisation implies that  $B \otimes A^n$  is a right  $B$ -module via application of  $\Psi$   $n$ -times.  $A^n$  here denotes the  $n$ -fold tensor product of  $A$ . This in turn implies that  $B \otimes A^n$  is an  $(X, X)$ -bimodule with the following structure maps. Left action is obtained by viewing  $B \otimes A^n$  as  $X \otimes A^{n-1}$  and multiplying from the left by elements of  $X$ . The right action is obtained by viewing  $X$  as  $B \otimes A$  and acting on  $B \otimes A^n$  by  $B$  as described above and then multiplying last factor by elements in  $A$ . Similarly (by interchanging  $A$  with  $B$  and “left” with “right”), one makes  $B^n \otimes A$  into a left  $A$ -module and then an  $(X, X)$ -bimodule. Using this bimodule structure of  $B \otimes A^n$  and  $B^n \otimes A$  one constructs the cohomology of the factorisation  $X(B, A)$  as follows.

First recall that the bar resolution of an algebra  $A$  is a chain complex  $\text{Bar}(A) = (\text{Bar}_\bullet(A), \delta_A)$ , where

$$\text{Bar}_n(A) = A^{n+2}, \quad \delta_{A,n} = \sum_{k=0}^n (-1)^k A^k \otimes \mu_A \otimes A^{n-k} : A^{n+2} \rightarrow A^{n+1}.$$

Consider bar resolutions of  $A$  and  $B$ . Apply functor  $- \otimes A$  to  $\text{Bar}(B)$  and the functor  $B \otimes -$  to  $\text{Bar}(A)$ . Since the definition of a bar complex boundary operator

involves the product in the algebra only one easily finds that both  $\text{Bar}(B)_\bullet \otimes A$ ,  $B \otimes \text{Bar}(A)_\bullet$  are  $(X, X)$ -bimodules and  $\delta_B \otimes A$ ,  $B \otimes \delta_A$  are bimodule maps. This implies that for any  $(X, X)$ -bimodule  $M$  there is a double cochain complex

$$C(X(B, A), M) = {}_X\text{Hom}_X((B \otimes \text{Bar}(A)) \otimes_X (\text{Bar}(B) \otimes A), M).$$

Explicitly, the space of  $(m, n)$ -cochains is

$$C^{m,n}(X(B, A), M) = {}_X\text{Hom}_X((B \otimes A^{m+2}) \otimes_X (B^{n+2} \otimes A), M) \cong \text{Hom}(A^m \otimes B^n, M).$$

This last identification is obtained as follows: for each  $\phi \in {}_X\text{Hom}_X((B \otimes A^{m+2}) \otimes_X (B^{n+2} \otimes A), M)$  one defines  $f_\phi \in \text{Hom}(A^m \otimes B^n, M)$  via

$$f_\phi(a_m, \dots, a_1, b^1, \dots, b^n) = \phi(1_B, 1_A, a_m, \dots, a_1, 1_A, 1_B, b^1, \dots, b^n, 1_B, 1_A),$$

while for any  $f \in \text{Hom}(A^m \otimes B^n, M)$  one defines  $\phi_f \in {}_X\text{Hom}_X((B \otimes A^{m+2}) \otimes_X (B^{n+2} \otimes A), M)$  by

$$\begin{aligned} \phi_f(b, a_{m+1}, \dots, a_0, b^0, \dots, b^{n+1}, a) \\ = \sum_{(\nu)(\tilde{\nu})} (bb_{\nu_0 \dots \nu_{m+1}}^0 a_{m+1}^{\nu_{m+1}}) \cdot f(a_m^{\nu_m}, \dots, a_1^{\nu_1}, b_{\tilde{\nu}_1}^1, \dots, b_{\tilde{\nu}_n}^n) \cdot (b_{\tilde{\nu}_{n+1}}^{n+1} a_0^{\nu_0 \tilde{\nu}_1 \dots \tilde{\nu}_{n+1}} a). \end{aligned}$$

The multi-index notation used above refers to multiple applications of the twisting map  $\Psi$ , i.e.,

$$\sum_{(\nu)} b_{\nu_0 \dots \nu_n} \otimes a_n^{\nu_n} \otimes \dots \otimes a_1^{\nu_1} = (\Psi \otimes A^{n-1})(A \otimes \Psi \otimes A^{n-2}) \dots (A^{n-1} \otimes \Psi)(a_n \otimes \dots \otimes a_1 \otimes b),$$

etc. From the deformation theory point of view the case  $M = X$  is of the greatest interest, thus to this case we restrict our attention from now on. Consider the Hochschild complexes of  $A$  and  $B$ . Notice that the inclusions  $\text{Hom}(A^m, A) \hookrightarrow \text{Hom}(A^m, X)$  and  $\text{Hom}(B^n, B) \hookrightarrow \text{Hom}(B^n, X)$  given by  $f \mapsto 1_B \otimes f$  and  $g \mapsto g \otimes 1_A$  respectively are inclusions of cochain complexes. The complex obtained from  $C(X(B, A), X)$  by replacing  $C^{\bullet,0}(X(B, A), X)$  by the Hochschild complex of  $A$ , and  $C^{0,\bullet}(X(B, A), X)$  by the Hochschild complex of  $B$  is denoted by  $C(X(B, A))$ . Explicitly, one has the following double complex

$$\begin{array}{ccccccccc} B \oplus A & \xrightarrow{d_A} & \text{Hom}(A, A) & \xrightarrow{d_A} & \text{Hom}(A^2, A) & \xrightarrow{d_A} & \text{Hom}(A^3, A) & \xrightarrow{d_A} & \dots \\ \downarrow d_B & & \downarrow d_B & & \downarrow d_B & & \downarrow d_B & & \\ \text{Hom}(B, B) & \xrightarrow{d_A} & \text{Hom}(A \otimes B, X) & \xrightarrow{d_A} & \text{Hom}(A^2 \otimes B, X) & \xrightarrow{d_A} & \text{Hom}(A^3 \otimes B, X) & \xrightarrow{d_A} & \dots \\ \downarrow d_B & & \downarrow d_B & & \downarrow d_B & & \downarrow d_B & & \\ \text{Hom}(B^2, B) & \xrightarrow{d_A} & \text{Hom}(A \otimes B^2, X) & \xrightarrow{d_A} & \text{Hom}(A^2 \otimes B^2, X) & \xrightarrow{d_A} & \text{Hom}(A^3 \otimes B^2, X) & \xrightarrow{d_A} & \dots \\ \downarrow d_B & & \downarrow d_B & & \downarrow d_B & & \downarrow d_B & & \end{array}$$

The coboundary operators  $d_A : C^{m,n}(X(B, A)) \rightarrow C^{m+1,n}(X(B, A))$  and  $d_B : C^{m,n}(X(B, A)) \rightarrow C^{m,n+1}(X(B, A))$ ,  $m, n > 0$  are given explicitly by

$$\begin{aligned} d_A f(a_{m+1}, \dots, a_1, b^1, \dots, b^n) &= a_{m+1} f(a_m, \dots, a_1, b^1, \dots, b^n) \\ &+ \sum_{i=0}^{m-1} (-1)^{i+1} f(a_{m+1}, \dots, a_{m+1-i} a_{m-i}, \dots, a_1, b^1, \dots, b^n) \\ &+ (-1)^{m+1} \sum_{(\nu)} f(a_{m+1}, \dots, a_2, b_{\nu_1}^1, \dots, b_{\nu_n}^n) a_1^{\nu_1 \dots \nu_n}, \end{aligned} \quad (4)$$

and

$$\begin{aligned} d_B f(a_m, \dots, a_1, b^1, \dots, b^{n+1}) &= \sum_{(\nu)} b_{\nu_1 \dots \nu_m}^1 f(a_m^{\nu_m}, \dots, a_1^{\nu_1}, b^2, \dots, b^{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(a_m, \dots, a_1, b^1, \dots, b^i b^{i+1}, \dots, b^{n+1}) \\ &+ (-1)^{n+1} f(a_m, \dots, a_1, b^1, \dots, b^n) b^{n+1}. \end{aligned} \quad (5)$$

For  $n = 0$ ,  $d_A$  is the usual Hochschild coboundary, while the  $d_B$  are given by (5), provided one views  $\text{Hom}(A^m, A)$  inside  $\text{Hom}(A^m, X)$  first. Similarly, for  $m = 0$ ,  $d_B$  is the usual Hochschild coboundary, while the  $d_A$  are given by (4), provided one views  $\text{Hom}(B^n, B)$  inside  $\text{Hom}(B^n, X)$  first. The construction of the above complex implies immediately that  $d_A \circ d_B = d_B \circ d_A$ , so that one can combine the double complex into a complex  $(C^\bullet(X(B, A)), D)$ ,

$$C^m(X(B, A)) = \text{Hom}(A^n, A) \oplus \bigoplus_{k=1}^{n-1} \text{Hom}(A^{n-k} \otimes B^k, X) \oplus \text{Hom}(B^n, B),$$

$D|_{C^{m,n}} = (-1)^m d_B + d_A$ . The cohomology of the complex  $(C^\bullet(X(B, A)), D)$  is denoted by  $H^\bullet(X(B, A))$ .

### 3 Cohomological interpretation of deformations

Two deformations  $X_t(B_t, A_t)$  and  $\tilde{X}_t(\tilde{B}_t, \tilde{A}_t)$  of an algebra factorisation  $X(B, A)$  are said to be *equivalent* to each other if there exist algebra isomorphisms  $\alpha_t : A_t \rightarrow \tilde{A}_t$ ,  $\beta_t : B_t \rightarrow \tilde{B}_t$  of the form  $\alpha_t = A + \sum_{i=1} t^i \alpha^{(i)}$ ,  $\beta_t = B + \sum_{i=1} t^i \beta^{(i)}$ , and such that  $\beta_t \otimes \alpha_t : X_t \rightarrow \tilde{X}_t$  is an algebra isomorphism. A deformation  $X_t(B_t, A_t)$  is called a *trivial deformation* if it is equivalent to an algebra factorisation in which all the maps  $\mu_A^{(i)}$ ,  $\mu_B^{(i)}$ ,  $\Psi^{(i)}$  in (3) vanish. An *infinitesimal deformation* of  $X(B, A)$  is a deformation of  $X(B, A)$  modulo  $t^2$ .

**Theorem 3.1** *There is a one-to-one correspondence between the equivalence classes of infinitesimal deformations of  $X(B, A)$  and  $H^2(X(B, A))$ .*

*Proof.* For an infinitesimal deformation it is enough to consider  $\mu_{A_t} = \mu_A + t\mu_A^{(1)}$ ,  $\mu_{B_t} = \mu_B + t\mu_B^{(1)}$ ,  $\Psi_t = \Psi + t\Psi^{(1)}$  where  $\mu_A^{(1)} \oplus \Psi^{(1)} \oplus \mu_B^{(1)} \in C^{2,0} \oplus C^{1,1} \oplus C^{0,2} = C^2(X(B, A))$ . First we show that the triple  $(\mu_A^{(1)}, \Psi^{(1)}, \mu_B^{(1)})$  defines an infinitesimal deformation if and only if  $\mu_A^{(1)} \oplus \Psi^{(1)} \oplus \mu_B^{(1)}$  is a cocycle.

As the first row and the first column in  $C(X(B, A))$  are simply Hochschild complexes, a standard algebra deformation theory argument shows that the associativity of  $\mu_{A_t}$  and  $\mu_{B_t}$  modulo  $t^2$  is equivalent to the conditions  $d_A\mu_A^{(1)} = d_B\mu_B^{(1)} = 0$ . In view of this fact we need to show that  $\Psi_t$  satisfies conditions (1) and (2) if and only if  $d_B\mu_A^{(1)} + d_A\Psi^{(1)} = 0$  and  $d_A\mu_B^{(1)} - d_B\Psi^{(1)} = 0$ . Expanding (1), (2) for  $\Psi_t$  in powers of  $t$  one easily finds that the  $t^0$ -order terms are simply equations (1), (2) for  $\Psi$ . Therefore only terms of order  $t$  require further study. The  $t$ -order term in the first of equations (1) is

$$(B \otimes \mu_A)(\Psi^{(1)} \otimes A)(A \otimes \Psi) - \Psi^{(1)}(\mu_A \otimes B) + (B \otimes \mu_A)(\Psi \otimes A)(A \otimes \Psi^{(1)}) - \Psi(\mu_A^{(1)} \otimes B) + (B \otimes \mu_A^{(1)})(\Psi \otimes A)(A \otimes \Psi) = 0.$$

This is precisely the statement that  $d_B\mu_A^{(1)} + d_A\Psi^{(1)} = 0$ . Evaluating this condition at  $1_A \otimes 1_A \otimes b$  one easily finds that  $\Psi^{(1)}(1_A \otimes b) = -b \otimes \mu_A^{(1)}(1_A \otimes 1_A) + \Psi(\mu_A^{(1)}(1_A \otimes 1_A) \otimes b)$ , i.e.,  $\Psi_t(1_{A_t} \otimes b) = b \otimes 1_{A_t}$ , where  $1_{A_t} = 1_A - \mu_A^{(1)}(1_A \otimes 1_A)t$  is the unit in the infinitesimal deformation  $A_t$ . Thus the second of equations (1) holds. Similarly one shows that the equations (2) hold for  $\Psi_t$  modulo  $t^2$  if and only if  $d_A\mu_B^{(1)} - d_B\Psi^{(1)} = 0$ . Therefore the necessary and sufficient condition for  $X(B, A)_t$  to be an infinitesimal deformation of  $X(B, A)$  is that  $\mu_A^{(1)} \oplus \Psi^{(1)} \oplus \mu_B^{(1)}$  be a 2-cocycle in  $C^2(X(B, A))$  as required.

Let  $X_t(B_t, A_t)$  and  $\tilde{X}_t(\tilde{B}_t, \tilde{A}_t)$  be two infinitesimal deformations of an algebra factorisation  $X(B, A)$  given by the cocycles  $\mu_A^{(1)} \oplus \Psi^{(1)} \oplus \mu_B^{(1)}$  and  $\tilde{\mu}_A^{(1)} \oplus \tilde{\Psi}^{(1)} \oplus \tilde{\mu}_B^{(1)}$  respectively. We need to show that these two deformations are equivalent to each other modulo  $t^2$  if and only if the corresponding cocycles differ by a coboundary. In view of the Gerstenhaber theory,  $\alpha_t = A + t\alpha : A_t \rightarrow \tilde{A}_t$  and  $\beta_t = B + t\beta : B_t \rightarrow \tilde{B}_t$  are the algebra isomorphisms modulo  $t^2$  if and only if  $\mu_A^{(1)} - \tilde{\mu}_A^{(1)} = d_A\alpha$  and  $\mu_B^{(1)} - \tilde{\mu}_B^{(1)} = d_B\beta$ . Thus it remains to be shown that  $\beta_t \otimes \alpha_t : X_t \rightarrow \tilde{X}_t$  is an algebra isomorphism modulo  $t^2$  if and only if

$$\Psi^{(1)} - \tilde{\Psi}^{(1)} = d_A\beta - d_B\alpha.$$

Suppose that  $\phi_t = \beta_t \otimes \alpha_t$  is an isomorphism of algebras, and let  $\phi$  be the  $t$ -order term in the expansion of  $\phi_t$ , i.e.,

$$\phi = B \otimes \alpha + \beta \otimes A. \quad (6)$$

Since  $\phi_t$  is an algebra map

$$\phi_t(ab) = \phi_t(a)\phi_t(b), \quad \forall a \in A, b \in B. \quad (7)$$

Note that the product on the left hand side of (7) is in  $X_t(B_t, A_t)$  while on the right hand side is in  $\tilde{X}_t(\tilde{B}_t, \tilde{A}_t)$ . One can use (6) to find the  $t$ -order term on the left hand side of (7)

$$\sum_{\nu} \phi(b_{\nu} a^{\nu}) + \Psi^{(1)}(a \otimes b) = \sum_{\nu} b_{\nu} \alpha(a^{\nu}) + \sum_{\nu} \beta(b_{\nu}) a^{\nu} + \Psi^{(1)}(a \otimes b).$$

All the products are in  $X(B, A)$  now. On the other hand the  $t$ -order term on the right hand side is

$$\tilde{\Psi}^{(1)}(a \otimes b) + a\phi(b) + \phi(a)b = \tilde{\Psi}^{(1)}(a \otimes b) + a\beta(b) + \alpha(a)b.$$

Thus if  $\phi_t$  is an algebra map we have

$$\begin{aligned} (\Psi^{(1)} - \tilde{\Psi}^{(1)})(a \otimes b) &= a\beta(b) - \sum_{\nu} \beta(b_{\nu}) a^{\nu} + \alpha(a)b - \sum_{\nu} b_{\nu} \alpha(a^{\nu}) \\ &= (d_A \beta - d_B \alpha)(a \otimes b), \end{aligned}$$

as required.

To prove the converse one needs to repeat the same computations in reversed order.  $\square$

The next step usually undertaken in the deformation theory, is to study obstructions for extending a deformation modulo  $t^n$  to a deformation modulo  $t^{n+1}$ . Such an obstruction consists of four terms. The first two terms come from the deformation of algebra structures of  $A$  and  $B$ , one for each algebra. They are:

$$\begin{aligned} \text{Obs}_A^{(n)} &= \sum_{k=1}^{n-1} [\mu_A^{(k)}(\mu_A^{(n-k)} \otimes A) - \mu_A^{(k)}(A \otimes \mu_A^{(n-k)})], \\ \text{Obs}_B^{(n)} &= \sum_{k=1}^{n-1} [\mu_B^{(k)}(\mu_B^{(n-k)} \otimes B) - \mu_B^{(k)}(B \otimes \mu_B^{(n-k)})], \end{aligned}$$

The remaining two obstructions arise from the factorisation conditions (1) and (2):

$$\begin{aligned}\text{Obs}_{A,\Psi}^{(n)} &= \sum_{k=1}^{n-1} \Psi^{(n-k)}(\mu_A^{(k)} \otimes B) - \sum_{\substack{k,l=0 \\ 1 \leq k+l \leq n}}^{n-1} (B \otimes \mu_A^{(k)})(\Psi^{(l)} \otimes A)(A \otimes \Psi^{(n-k-l)}), \\ \text{Obs}_{B,\Psi}^{(n)} &= \sum_{\substack{k,l=0 \\ 1 \leq k+l \leq n}}^{n-1} (\mu_B^{(k)} \otimes A)(B \otimes \Psi^{(l)})(\Psi^{(n-k-l)} \otimes B) - \sum_{k=1}^{n-1} \Psi^{(n-k)}(A \otimes \mu_B^{(k)}).\end{aligned}$$

Here  $\Psi^{(0)} = \Psi$ ,  $\mu_A^{(0)} = \mu_A$  and  $\mu_B^{(0)} = \mu_B$ . The following theorem is an algebra factorisation version of a standard result in the deformation theory.

**Theorem 3.2** *If  $X_t(B_t, A_t)$  is a deformation of  $X(B, A)$  modulo  $t^n$  then*

$$\text{Obs}^{(n)} = \text{Obs}_A^{(n)} \oplus \text{Obs}_{A,\Psi}^{(n)} \oplus \text{Obs}_{B,\Psi}^{(n)} \oplus \text{Obs}_B^{(n)}$$

*is a 3-cocycle in the complex  $C(X(B, A))$ .  $X_t(B_t, A_t)$  can be extended to a deformation of  $X(B, A)$  modulo  $t^{n+1}$  if and only if  $\text{Obs}^{(n)}$  is a coboundary.*

*Proof.* The first part of the theorem can be proven in the following way (standard in the deformation theory of algebras, which also asserts that  $\text{Obs}_A^{(n)}$  and  $\text{Obs}_B^{(n)}$  are Hochschild cocycles). Let

$$\tilde{\mu}_A = \mu_A + \sum_{i=1}^{n-1} t^i \mu_A^{(i)}, \quad \tilde{\mu}_B = \mu_B + \sum_{i=1}^{n-1} t^i \mu_B^{(i)}, \quad \tilde{\Psi} = \Psi + \sum_{i=1}^{n-1} t^i \Psi^{(i)}.$$

The proof hinges on two observations. Firstly, one easily finds that

$$\begin{aligned}\text{Obs}_A^{(n)} &= \text{coefficient of } t^n \text{ in } \tilde{\mu}_A(\tilde{\mu}_A \otimes A) - \tilde{\mu}_A(A \otimes \tilde{\mu}_A), \\ \text{Obs}_{A,\Psi}^{(n)} &= \text{coefficient of } t^n \text{ in } \tilde{\Psi}(\tilde{\mu}_A \otimes B) - (B \otimes \tilde{\mu}_A)(\tilde{\Psi} \otimes A)(A \otimes \tilde{\Psi}), \\ \text{Obs}_{B,\Psi}^{(n)} &= \text{coefficient of } t^n \text{ in } (\tilde{\mu}_A \otimes B)(B \otimes \tilde{\Psi})(\tilde{\Psi} \otimes B) - \tilde{\Psi}(A \otimes \tilde{\mu}_B), \\ \text{Obs}_B^{(n)} &= \text{coefficient of } t^n \text{ in } \tilde{\mu}_B(\tilde{\mu}_B \otimes B) - \tilde{\mu}_B(B \otimes \tilde{\mu}_B).\end{aligned}$$

Secondly one should notice that

$$D\text{Obs}^{(n)} = \text{coefficient of } t^n \text{ in } \tilde{D}\text{Obs}^{(n)},$$

where  $\tilde{D}$  is obtained by replacing  $\mu_A$ ,  $\mu_B$  and  $\Psi$  in definition of  $D$  with  $\tilde{\mu}_A$ ,  $\tilde{\mu}_B$  and  $\tilde{\Psi}$ . Expanding  $\tilde{D}\text{Obs}^{(n)}$ , with  $\text{Obs}^{(n)}$  expressed entirely in terms of the tilded structure



maps, one discovers that the term-by-term cancellations yield  $\tilde{D}\text{Obs}^{(n)} = 0$ . Thus  $\text{Obs}^{(n)}$  is a cocycle as asserted. (This expansion is a straightforward procedure, one only has to remember to take the inclusions of Hochschild cocycles into  $C(X(B, A))$  properly into account.)

It follows from the Gerstenhaber theory that  $A_t$  and  $B_t$  are deformations of  $A$  and  $B$  respectively modulo  $t^{n+1}$  if and only if  $\text{Obs}_A^{(n)}$  and  $\text{Obs}_B^{(n)}$  are coboundaries in the Hochschild cohomology, i.e., there exist  $\mu_A^{(n)} : A \otimes A \rightarrow A$  and  $\mu_B^{(n)} : B \otimes B \rightarrow B$  such that  $d_A \mu_A^{(n)} = \text{Obs}_A^{(n)}$  and  $d_B \mu_B^{(n)} = \text{Obs}_B^{(n)}$ . Thus only the conditions arising from (1) and (2) require further study. Gathering all the terms of order  $t^n$  in (1) and (2) one easily finds that  $X_t(B_t, A_t)$  is a deformation modulo  $t^{n+1}$  if and only if

$$d_A \Psi^{(n)} + d_B \mu_A^{(n)} = \text{Obs}_{A, \Psi}^{(n)}, \quad d_A \mu_B^{(n)} - d_B \Psi^{(n)} = \text{Obs}_{B, \Psi}^{(n)}.$$

All this means that the necessary and sufficient condition for  $X_t(B_t, A_t)$  to be a deformation modulo  $t^{n+1}$  is that

$$D(\mu_A^{(n)} \oplus \Psi^{(n)} \oplus \mu_B^{(n)}) = \text{Obs}^{(n)},$$

i.e.,  $\text{Obs}^{(n)}$  is a coboundary, as required.  $\square$

An interesting special case of this general deformation theory is a deformation  $X_t(B, A)$ , i.e., the algebras  $B$  and  $A$  are not deformed, and only the formal power series  $\Psi_t$  is non-trivial. This type of deformation is considered in [4]. In this case  $\text{Obs}_A^{(n)} = 0$ ,  $\text{Obs}_B^{(n)} = 0$ , and

$$\begin{aligned} \text{Obs}_{A, \Psi}^{(n)} &= - \sum_{i=1}^{n-1} (B \otimes \mu_A)(\Psi^{(i)} \otimes A)(A \otimes \Psi^{(n-i)}), \\ \text{Obs}_{B, \Psi}^{(n)} &= \sum_{i=1}^{n-1} (\mu_B \otimes A)(B \otimes \Psi^{(i)})(\Psi^{(n-i)} \otimes B). \end{aligned}$$

The obstruction removing equations are:

$$d_A \Psi^{(n)} = \text{Obs}_{A, \Psi}^{(n)}, \quad d_B \Psi^{(n)} = -\text{Obs}_{B, \Psi}^{(n)},$$

and coincide with the equations given in [4, Theorem 4.11].

We would like to conclude with three concrete examples illustrating the deformation theory of algebra factorisations. The first example deals with a deformation affecting both the algebra structure of  $A$  as well as the map  $\Psi$ , while the remaining two are an illustration of a deformation of  $\Psi$  only.

**Example 3.3** Let  $k = \mathbf{C}$ ,  $A = \mathbf{C}[a, \bar{a}]/(a\bar{a} - \bar{a}a)$  (an algebra of polynomials in two commuting variables),  $B = \mathbf{C}[b]$ , and  $X = B \otimes A$  a tensor product algebra (an algebra of polynomials in three commuting variables).  $A$  is spanned by all monomials  $a^k \bar{a}^l$ ,  $k, l = 0, 1, 2, \dots$ , while  $B$  is spanned by the set  $\{b^r \mid r = 0, 1, 2, \dots\}$ . Thus the structure maps are  $\mu_A(a^k \bar{a}^l \otimes a^r \bar{a}^s) = a^{k+r} \bar{a}^{l+s}$ ,  $\mu_B(b^r \otimes b^s) = b^{r+s}$ , while the map  $\Psi$  is the usual twist,  $\Psi(a^k \bar{a}^l \otimes b^r) = b^r \otimes a^k \bar{a}^l$ . One easily verifies that  $\mu_A^{(1)} \oplus \Psi^{(1)}$ , where

$$\Psi^{(1)}(a^k \bar{a}^l \otimes b^r) = lr b^r \otimes a^k \bar{a}^l - kr b^{r+1} \otimes a^{k-1} \bar{a}^{l+1}, \quad \mu_A^{(1)}(a^k \bar{a}^l \otimes a^r \bar{a}^s) = lr a^{k+r} \bar{a}^{l+s},$$

is a cocycle in  $C^\bullet(X(B, A))$  and therefore defines an infinitesimal deformation of  $X(B, A)$ . The  $n = 2$  obstruction 3-cocycle consists of three terms:

$$\begin{aligned} \text{Obs}_A^{(2)}(a^k \bar{a}^l \otimes a^m \bar{a}^n \otimes a^p \bar{a}^r) &= lp(lm - np)a^{k+m+p} \bar{a}^{l+n+r}, \\ \text{Obs}_{A, \Psi}^{(2)}(a^k \bar{a}^l \otimes a^m \bar{a}^n \otimes b^r) &= r((lm + kn)r + km)b^{r+1} \otimes a^{k+m-1} \bar{a}^{l+n+1} \\ &\quad - lnr^2 b^r \otimes a^{k+m} \bar{a}^{l+n} - kmr(r+1)b^{r+2} \otimes a^{k+m-2} \bar{a}^{l+n+2}, \\ \text{Obs}_{B, \Psi}^{(2)}(a^k \bar{a}^l \otimes b^r \otimes b^s) &= l^2 r s b^{r+s} \otimes a^k \bar{a}^l - (2l+1)krs b^{r+s+1} \otimes a^{k-1} \bar{a}^{l+1} \\ &\quad + (k-1)krs b^{r+s+2} \otimes a^{k-2} \bar{a}^{l+2}. \end{aligned}$$

This obstruction can be removed by setting

$$\begin{aligned} \mu_A^{(2)}(a^k \bar{a}^l \otimes a^m \bar{a}^n) &= lm(c + \frac{lm}{2})a^{k+m} \bar{a}^{l+n} \\ \Phi^{(2)}(a^k \bar{a}^l \otimes b^r) &= lr(c + \frac{lr}{2})b^r \otimes a^k \bar{a}^l - kr(\frac{k+r-1}{2} + lr + c)b^{r+1} \otimes a^{k-1} \bar{a}^{l+1} \\ &\quad - \frac{1}{2}k(k-1)r(r+1)b^{r+2} \otimes a^{k-2} \bar{a}^{l+2}, \end{aligned}$$

where  $c$  is a constant number. This deformation of  $X(B, A)$  modulo  $t^3$  has presentation with generators  $a, \bar{a}$  and  $b$ , and the relations

$$\bar{a}a = qa\bar{a}, \quad \bar{a}b = qb\bar{a}, \quad ab = ba + (1-q)b^2\bar{a},$$

where  $q = 1 + t + (c + \frac{1}{2})t^2$ . A simple calculation reveals that the above relations define an associative algebra to all powers of  $t$ . Furthermore the same calculation implies that the above relations with  $q = 1 + t + (c + \frac{1}{2})t^2 + o(t^3)$  describe an associative algebra factorisation  $X_t(B, A_t)$  over  $\mathbf{C}[[t]]$ . The deformed product in  $A$  and deformed twisting map come out as:

$$\mu_{A_t}(a^k \bar{a}^l \otimes a^r \bar{a}^s) = q^{lr} a^{k+r} \bar{a}^{l+s},$$

$$\Psi_t(a^k \bar{a}^l \otimes b^r) = q^{lr} \sum_{i=0}^k \binom{k}{i}_q \binom{r+i-1}{i}_q (q; q)_i b^{r+i} \otimes a^{k-i} \bar{a}^{l+i},$$

where  $(q; q)_{-1} = (q; q)_0 = 1$ ,  $(q; q)_i = (1-q)(1-q^2) \cdots (1-q^i)$ ,

$$\binom{k}{i}_q = \frac{(q; q)_k}{(q; q)_i (q; q)_{k-i}}.$$

**Example 3.4** Let  $k = \mathbf{C}$ ,  $A = \mathbf{C}[a, \bar{a}]/(\bar{a}a - qa\bar{a})$  (Manin's quantum plane),  $B = \mathbf{C}[b]$  and  $\Psi : A \otimes B \rightarrow B \otimes A$ ,  $\Psi(a^k \bar{a}^l \otimes b^r) = q^{lr} b^r \otimes a^k \bar{a}^l$ , where  $q$  is a non-zero complex number which is not a root of unity. The resulting algebra factorisation is

$$X(B, A) = \mathbf{C}[b, a, \bar{a}]/(\bar{a}a - qa\bar{a}, \bar{a}b - qb\bar{a}, ab - ba).$$

One easily verifies that  $\Psi^{(1)} : A \otimes B \rightarrow B \otimes A$ ,

$$\Psi^{(1)}(a^k \bar{a}^l \otimes b^r) = q^{lr} [k][r] b^{r+1} \otimes a^{k-1} \bar{a}^{l+1},$$

where  $[k] = \frac{1-q^k}{1-q}$ , is a cocycle in  $C^\bullet(X(B, A))$  and therefore defines an infinitesimal deformation of  $X(B, A)$ . This infinitesimal deformation of  $X(B, A)$  has a presentation with generators  $a, \bar{a}, b$  and relations:

$$\bar{a}a = qa\bar{a}, \quad \bar{a}b = qb\bar{a}, \quad ab = ba + tb^2\bar{a}. \quad (8)$$

The infinitesimal deformation given by relations (8) can be extended to a deformation to all orders in  $t$  by setting:

$$\Psi^{(i)}(a^k \bar{a}^l \otimes b^r) = q^{lr} [k][k-1] \cdots [k-i+1] \binom{r+i-1}{i}_q b^{r+i} \otimes a^{k-i} \bar{a}^{l+i},$$

with the same notation as in Example 3.3 (but note that  $q$  has a different meaning). The resulting algebra factorisation  $X_t(B, A)$  over  $\mathbf{C}[[t]]$  has presentation with the relations (8). The fact that with these definitions of the  $\Psi^{(i)}$  all the obstructions are removed can be verified directly by using various identities for  $q$ -binomial coefficients. This can also be verified by checking directly that relations (8) define an associative algebra.

**Example 3.5** *Deformation of quaternions.* Let  $k = \mathbf{R}$ ,  $A = \mathbf{R}[i]/(i^2 + 1) \cong \mathbf{C}$ ,  $B = \mathbf{R}[j]/(j^2 + 1) \cong \mathbf{C}$ , and set  $\Psi(i \otimes j) = -j \otimes i$ . The resulting algebra is  $X = \mathbf{H}$ , i.e., there is a factorisation  $\mathbf{H}(\mathbf{C}, \mathbf{C})$ . It is well known that the second Hochschild cohomology of  $\mathbf{C}$  viewed as a real algebra is trivial. Thus we can choose  $\mu_A^{(1)} = 0$  and  $\mu_B^{(1)} = 0$ . This implies that any cocycle in  $C^2(\mathbf{H}(\mathbf{C}, \mathbf{C}))$  must be cohomology equivalent to the map  $\Psi^{(1)} : \mathbf{C} \otimes \mathbf{C} \rightarrow \mathbf{C} \otimes \mathbf{C}$  such that  $\Psi^{(1)}(1 \otimes j) = \Psi^{(1)}(i \otimes 1) = \Psi^{(1)}(1 \otimes 1) = 0$ . All such cocycles can be easily computed, and one finds that  $\dim H^2(\mathbf{H}(\mathbf{C}, \mathbf{C})) = 1$  with the unique cohomology class generated by the cocycle  $\Psi^{(1)}(i \otimes j) = 1 \otimes 1$ . Consequently, there is only one deformation of quaternions that retains the factorisation property, namely the algebra factorisation  $\mathbf{H}_t(\mathbf{C}, \mathbf{C})$  spanned by  $1, i, j$  and  $ij$  subject to the relations  $i^2 = j^2 = -1$ ,  $ij + ji = t$ . This result agrees with the classification of factorisations given in [3, Example 2.12].

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